

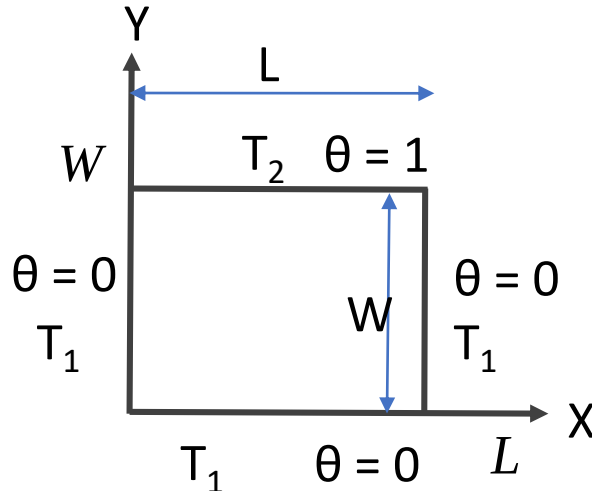
Heat and Mass Transfer

2-D Conduction: Analytical Solution

Steady state 2-D Conduction: Analytical Solution

Let us consider a rectangular plate of length L and width W . Three sides of the Plate are maintained at a temperature T_1 while the fourth side is maintained at a temperature T_2 ($T_2 \neq T_1$). We require to find out the temperature distribution on the plate.

We define
$$\theta = \frac{T - T_1}{T_2 - T_1}$$



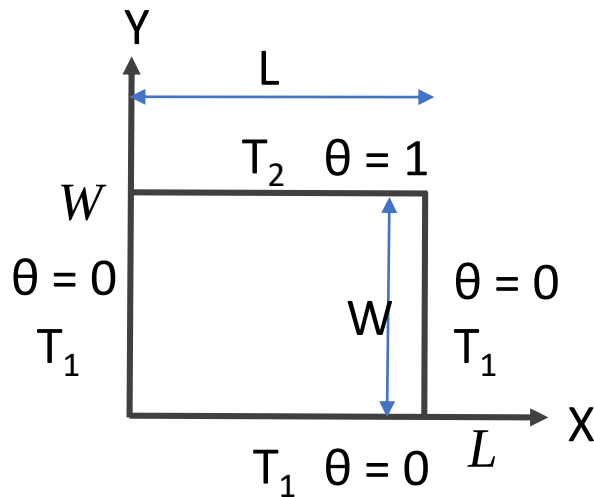
Boundary Conditions:

$$\theta(0, y) = 0 \quad 0 < y < W$$

$$\theta(L, y) = 0 \quad 0 < y < W$$

$$\theta(x, 0) = 0 \quad 0 < x < L$$

$$\theta(x, W) = 1 \quad 0 < x < L$$



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Existence of a solution: $\theta = X(x) Y(y)$

Separation of variables, solution is the product two functions: $\theta = X(x) Y(y)$

GE:
$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

$$-\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = \lambda^2$$

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0$$

$$\frac{\partial^2 Y}{\partial y^2} - \lambda^2 Y = 0$$

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{\lambda y}$$

$$\theta = (C_1 \cos \lambda x + C_2 \sin \lambda x)(C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

Boundary Condition-1: $x = 0, \quad \theta(x = 0, y) = 0, \quad 0 < y < W$

$$C_1(C_3 e^{-\lambda y} + C_4 e^{\lambda y}) = 0$$

Gives us

$$C_1 = 0$$

Therefore,

$$\theta = C_2 \sin \lambda x (C_3 e^{-\lambda y} + C_4 e^{\lambda y})$$

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Boundary Condition-3: $Y = 0, \quad \theta(x, 0) = 0, \quad 0 < x < L$

$$C_2 \sin \lambda x (C_3 + C_4) = 0$$

If $C_2 = 0$, this equality will eliminate x dependence and will produce Unacceptable solution. So only possible criterion is

$$(C_3 + C_4) = 0 \qquad C_3 = -C_4$$

$$\theta = C_2 \sin \lambda x \{C_4 (-e^{-\lambda y} + e^{\lambda y})\}$$

Boundary Condition-2: $x = L \quad \theta(L, y) = 0 \quad 0 < y < W$

$$C_2 C_4 \sin \lambda L (-e^{-\lambda y} + e^{\lambda y}) = 0$$

The only way in which this condition may be satisfied is by requiring that λ assumes discrete values for which $\sin \lambda L = 0$

These values then must have the form, $\lambda = n\pi/L$ ($n = 1, 2, 3, \dots$ any integer

$$\theta = C_2 C_4 \sin \frac{n\pi x}{L} (-e^{-\lambda y} + e^{\lambda y})$$

$n = 0$ is not included (then θ will be independent of y)

$$\theta = \sum_{n=1,2,3}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

In order to determine C_n we apply the fourth boundary condition which is

Boundary Condition-4: at $y = W$, $\theta(x, W) = 1$ for $0 \leq x \leq L$

$$\sum_{n=1,2,3}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi W}{L} = 1 \quad \text{Let,} \quad \frac{n\pi x}{L} = \lambda_n x$$

$$\int_0^L \left(\sum_{n=1,2,3}^{\infty} C_n \sin \lambda_n x \sinh \lambda_n W \right) \sin \lambda_m x \, dx = \int_0^L \sin \lambda_m x \, dx$$

$$\left(\sum_{n=1,2,3}^{\infty} C_n \sinh \lambda_n W \right) = \frac{\int_0^L \sin \lambda_n x \, dx}{\int_0^L \sin^2 \lambda_n x \, dx} = A_n$$

$$\int_a^b \sin \lambda_n x \sin \lambda_m x \, dx = 0 \text{ for } n \neq m$$

$$\neq 0 \text{ for } n = m$$

$$A_n = \frac{2}{n\pi} [(-1)^{n+1} + 1]$$

$$\int_0^L \sin \frac{n\pi x}{L} dx = \frac{L}{n\pi} \left[-\cos \frac{n\pi x}{L} \right]_0^L = \frac{L}{n\pi} (1 - \cos n\pi) = \frac{L}{n\pi} [1 + (-1)^{n+1}]$$

$$\int_0^L \sin^2 \lambda_n x dx = \int_0^L \frac{1}{2} (1 - \cos 2\lambda_n x) dx = \frac{1}{2} \left[x - \frac{1}{2\lambda_n} \sin 2\lambda_n x \right]_0^L$$

$$= \frac{1}{2} \left[x - \frac{L}{2n\pi} \sin \frac{2n\pi x}{L} \right]_0^L = \frac{L}{2}$$

Therefore,

$$A_n = \frac{2}{n\pi} [(-1)^{n+1} + 1]$$

The properties of orthogonal functions are used to solve $\sum_{n=1,2,3}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi W}{L} = 1$

for C_n by formulating an analogous infinite series for the appropriate form

Now we can write

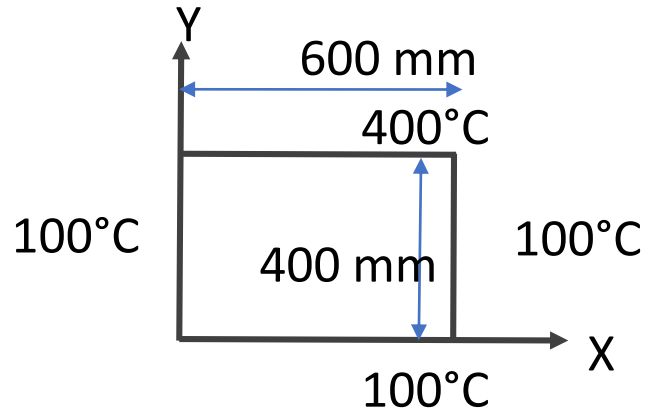
$$\sum_{n=1,2,3}^{\infty} C_n = \frac{2}{\pi} \sum_{n=1,2,3}^{\infty} \frac{(-1)^{n+1} + 1}{n} \frac{1}{\sinh(n\pi W/L)}$$

Therefore, the complete solution is

$$\theta = \sum_{n=1,2,3}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L}$$

$$\theta(x, y) = \frac{2}{\pi} \sum_{n=1,2,3}^{\infty} \frac{(-1)^{n+1} + 1}{n} \sin \frac{n\pi x}{L} \left\{ \frac{\sinh(n\pi y/L)}{\sinh(n\pi W/L)} \right\}$$

Example: Find the temperature at a point $x=0.3$ m and $y=0.2$ m on a plate



$$T_1 = 100^\circ\text{C} \quad T_2 = 400^\circ\text{C}$$

$k = 50 \text{ W/mK}$, use $n = 1$ and 3

$$T_p(300, 200) = ?$$

$$x_p = 0.3 \text{ m}, y_p = 0.2 \text{ m}$$

$$L = 0.6 \text{ m}, W = 0.4 \text{ m}$$

Taking first two terms in the series

$$\frac{T - T_1}{T_2 - T_1} = \frac{4}{\pi} \left[\frac{\sin \frac{\pi x_p}{L} \sinh \frac{\pi y_p}{L}}{\sinh \frac{\pi W}{L}} + \frac{\sin \frac{3\pi x_p}{L} \sinh \frac{3\pi y_p}{L}}{3 \sinh \frac{3\pi W}{L}} \right]$$

$$\sin \frac{\pi x_p}{L} = \sin \frac{\pi}{2} = 1$$

$$\sinh \frac{\pi y_p}{L} = \sinh \frac{\pi}{3} = \frac{e^{\pi/3} - e^{-\pi/3}}{2} = 1.25$$

$$\sin \frac{\pi W}{L} = \sin \frac{2\pi}{3} = 0.866$$

$$\sinh \frac{2\pi}{3} = 4.00$$

$$\sin \frac{3\pi x_p}{L} = \sin \frac{3\pi}{2} = -1$$

$$\sinh \frac{3\pi y_p}{L} = \sinh \pi = 11.54$$

$$\sinh \frac{3\pi W}{L} = \sinh \left(3\pi \cdot \frac{2}{3} \right) = 267.75$$

$$\frac{T_p - T_1}{T_2 - T_1} = \frac{4}{\pi} \left[\frac{1 \times 1.25}{4.00} + \frac{(-1) \times 11.54}{3 \times 267} \right]$$

$$\frac{T_p - 100}{400 - 100} = \frac{4}{\pi} [0.3125 - 0.0144]$$

$$T_p = 100 + 300 \times 0.3795 = 213.85^\circ\text{C}$$

Properties of Orthogonal Functions:

An infinite set of functions $g_1(x), g_2(x), \dots, g_n(x)$ is said to be orthogonal in the domain $a \leq x \leq b$ if

$$\int_a^b g_m(x) g_n(x) dx = 0 \quad \text{for } m \neq n \quad (1)$$

Many functions exhibit orthogonality including the trigonometric functions

$$\sin \frac{n\pi x}{L} \quad \text{and} \quad \cos \frac{n\pi x}{L} \quad \text{for } 0 \leq x \leq L$$

Their utility in the present problem rests with the fact that any function $f(x)$ may be expressed in terms of an infinite series of orthogonal functions

$$f(x) = \sum_{n=1}^{\infty} A_n g_n(x) \quad (2)$$

The form A_n in the series may be determined by multiplying each side of the equation by $g_n(x)$ and integrating between a and b

$$\int_a^b f(x)g_n(x) dx = \int_a^b g_n(x) \sum_{n=1}^{\infty} A_n g_n(x) dx \quad (3)$$

However, from (1) it is evident that all but one of the terms on the right-hand side of Eqn. (3) must be zero, leaving us with

$$\int_a^b f(x)g_n(x) dx = A_n \int_a^b g_n^2(x) dx$$

$$A_n = \frac{\int_a^b f(x)g_n(x) dx}{\int_a^b g_n^2(x) dx} \quad (4)$$

We will stop here today